SUBJECT: MATHEMATICS

SEMESTER-IV

PAPER-I, UNIT – I: DIFFERENTIAL EQUATIONS,

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METHOD OF REDUCTION:

(Whose one solution of complementary function is known)

If $y = u$ is given solution belonging to the complementary function of the differential equation.

Let the other solution be $y = v.$ Then $y = u.\, v$ is complete solution of the differential equation.

Let $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = R$ …..(1) be the differential equation and u is the complementary function of (1)

$$\therefore \qquad \frac{d^2u}{dx^2} + p\frac{du}{dx} + Qu = 0 \qquad \ldots(2)$$

$$y = u.\, v \text{ so that } \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + u\frac{d^2y}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we get

$$v\frac{d^2u}{dx^2} + 2\frac{dv}{dx}\frac{du}{dx} + u\frac{d^2v}{dx^2} + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Qu.v = R$$

On arranging

$$\Rightarrow \qquad v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + u\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right] + 2\frac{du}{dx}\cdot\frac{dv}{dx} = R$$

The first bracket is zero by virtue of relation (2), and the remaining is divided by u.

$$\frac{d^2v}{dx^2}+P\frac{dv}{dx}+\frac{2}{u}\frac{du}{dx}\frac{dv}{dx}=\frac{R}{u}$$

$\Rightarrow$ $$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=\frac{R}{u}$$

Let $\frac{dv}{dx}=z,$ so that $\frac{d^2v}{dx^2}=\frac{dz}{dx}$

Equation (3) becomes

$$\boxed{\frac{dz}{dx}+\left[P+\frac{2}{u}\frac{du}{dx}\right]z=\frac{R}{u}}$$

This is the linear differential equation of first order and can be solved (z can be found), which will contain one constant.

On integration $z=\frac{dv}{dx}$, we can get v .

Having found v , the solution is $y=uv.$

Ques. Solve $y''-4xy'+(4x^2-2)y=0$ given that $y=e^{x^2}$ is an integral included in the complementary function.

Sol. Here, we have $y''-4xy'+(4x^2-2)y=0$

On putting $y=v.e^{x^2}$ in (1), the reduced equations is

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=0 \qquad [P=-4x, Q=4x^2-2, R=0]$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-4x+\frac{2}{e^{x^2}}(2xe^{x^2})\right]\frac{dv}{dx}=0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2}+[-4x+4x]\frac{dv}{dx}=0 \quad \Rightarrow \quad \frac{d^2v}{dx^2}=0 \Rightarrow \frac{dv}{dx}=c_1 \Rightarrow v=c_1x+c_2$$

$\therefore$ $$y=uv \qquad [u=e^{x^2}]$$

$$y=e^{x^2}(c_1x+c_2)$$ Ans.

Ques. Solve $x^2y''-(x^2+2x)y'+(x+2)y=x^3e^x$ given that $y=x$ is *one* solution.

Sol. Here, we have $x^2y''-(x^2+2x)y'+(x+2)y=x^3e^x$

$$\Rightarrow \qquad y'' - \frac{x^2+2x}{x^2} y' + \frac{x+2}{x^2} y = x e^x \qquad \ldots(1)$$

On putting $y = vx$ in (1), the reduced equation is

$$\frac{d^2v}{dx^2} + \left\{P + \frac{2}{u}\frac{du}{dx}\right\}\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left[-\frac{x^2+2x}{x^2} + \frac{2}{x}(1)\right]\frac{dv}{dx} = \frac{xe^x}{x}$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x \quad \Rightarrow \quad \frac{dz}{dx} - z = e^x \qquad \left(\because z = \frac{dv}{dx}\right)$$

which is a linear differential equation

$$I.F. = e^{-\int dx} = e^{-x}$$

Its solution is $\qquad z e^x = \int e^x . e^{-x} dx + c$

$$\Rightarrow \qquad ze^{-x} = x + c \quad \Rightarrow \quad z = e^x . x + ce^x$$

$$\Rightarrow \qquad \frac{dv}{dx} = e^x . x + c e^x$$

$$\Rightarrow \qquad v = x.e^x - e^x + c e^x + c_1$$

$$\Rightarrow \qquad v = (x-1)e^x + c e^x + c_1$$

$$y = vx = (x^2 - x + cx)e^x + c_1 x$$ Ans.

Ques. Solve $\quad x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$

Given that $y = e^x$ is an integral included in the complementary function.

Sol. Here, we have $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} - \frac{2x-1}{x}\frac{dy}{dx} + \frac{x-1}{x} y = 0 \qquad \ldots(1)$$

By putting $y = ve^x$ in (1), we get the reduced equation as

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=0 \qquad \ldots(2)$$

Putting $u=e^x$ and $\frac{dv}{dx}=z$ in (2), we get

$$\frac{dz}{dx}+\left[-\frac{2x-1}{x}+\frac{2}{e^x}e^x\right]z=0$$

$$\Rightarrow \quad \frac{dz}{dx}+\frac{-2x-1+2x}{x}z=0 \qquad \Rightarrow \frac{dz}{dx}+\frac{z}{x}=0$$

$$\Rightarrow \quad \frac{dz}{z}=\frac{dx}{x} \quad \Rightarrow \quad \log z=-\log x+\log c$$

$$\Rightarrow \quad z=\frac{c_1}{x} \quad \Rightarrow \quad \frac{dv}{dx}=\frac{c_1}{x} \quad \Rightarrow \quad d\,v=c_1\frac{dx}{x} \quad \Rightarrow \quad c_1\log x+c_2$$

$$y=u.\,v=e^x(c_1\log x+c_2)$$

RULE TO FIND OUT PART OF THE COMPLEMENTARY FUNCTION

Rule	Condition	Part of Complementary Function $=u$
1	$1+P+Q=0$	e^x
2	$1-P+Q=0$	e^{-x}
3	$1+\frac{P}{a}+\frac{Q}{a^2}=0$	e^{ax}
4	$P+Qx=0$	x
5	$2+2Px+Qx^2=0$	x^2
6	$n(n-1)+Pnx+Qx^2=0$	x^n

Ques. Solve $x^2\frac{d^2y}{dx^2}-2x[1+x]\frac{dy}{dx}+2(1+x)y=x^3$

Sol. $x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-\frac{2x(1+x)}{x^2}\frac{dy}{dx}+\frac{2(1+x)y}{x^2}=x$$

Here, $$P+Qx=-\frac{2x(1+x)}{x^2}+\frac{2(1+x)}{x^2}x=0$$

Hence $y=x$ is a solution of the C.F. and the other solution is v.

Putting $y=v\,x$ in (1), we get the reduced equation as

$$\frac{d^2v}{dx^2}+\left\{P+\frac{2}{u}\frac{du}{dx}\right\}\frac{dv}{dx}=\frac{x}{u}$$

$$\frac{d^2v}{dx^2}+\left[\frac{-2x(1+x)}{x^2}+\frac{2}{x}(1)\right]\frac{dv}{dx}=\frac{x}{x}$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2}-2\frac{dv}{dx}=1 \quad\Rightarrow\quad \frac{dz}{dx}-2z=1 \qquad \left[\frac{dv}{dx}=z\right]$$

Which is a linear differential equation of first order and $I.F.=e^{\int -2dx}=e^{-2x}$

Its solution is $$z\,e^{-2x}=\int e^{-2x}dx+c_1$$

$$\Rightarrow \qquad ze^{-2x}=\frac{e^{-2x}}{-2}+c_1 \Rightarrow z=\frac{-1}{2}+c_1e^{2x}$$

$$\Rightarrow \qquad \frac{dv}{dx}=-\frac{1}{2}+c_1e^{2x} \quad\Rightarrow\quad dv=\left(-\frac{1}{2}+c_1e^{2x}\right)dx$$

$$\Rightarrow \qquad v=\frac{-x}{2}+\frac{c_1}{2}e^{2x}+c_2$$

$$y=uv=x\left(\frac{-x}{2}+\frac{c_1}{2}e^{2x}+c_2\right) \qquad \text{Ans.}$$

Method of reduction of order:

Method 1. To Find the Complete Solution of $y''+Py'+Qy=R$ when part of Complementary Function is known (Method of reduction of order)

Let $y=u$ be a part of the complementary function of the given differential equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad \ldots(1)$$

Where u is a function of x. Then, we have

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad \ldots(2)$$

Let $y = uv$ be the complete solution of equation (1), where v is a function of x.

Differentiating y w.r.t. x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}.v$$

Again, $$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}.\frac{du}{dx} + v\frac{d^2u}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1) we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Q(uv) = R$$

$$\Rightarrow \qquad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\Rightarrow \qquad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R \qquad \text{Using (2)}$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u} \qquad \ldots(3)$$

Put $\frac{dv}{dx} = p$ then, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

Now (3) becomes, $$\frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)p = \frac{R}{u} \qquad \ldots.(4)$$

Equation (4) is a linear differential equation of I order in p and x.

$$I.F. = e^{\int\left(\frac{2}{u}\frac{du}{dx} + p\right)dx} = e^{\left(\int\frac{2}{u}du + \int P\,dx\right)} = u^2\, e^{\int P\,dx}$$

Solution of (4) is given by

$$pu^2\, e^{\int P\,dx} = \int\frac{R}{u}u^2\, e^{\int P\,dx}dx + c_1$$

Where c_1 is an arbitrary constant of integration.

$\Rightarrow \qquad p = \frac{1}{u^2} e^{-\int P\,dx}\left[\int Ru\, e^{\int P\,dx} dx + c_1\right]$

$\therefore \qquad \frac{dv}{dx} = \frac{1}{u^2} e^{-\int P\,dx}\left[\int Ru\, e^{\int P\,dx} dx + c_1\right]$

Integration yields, $\quad v = \int \frac{1}{u^2} e^{-\int P\,dx}\left[\int Ru\, e^{\int P\,dx} dx + c_1\right] dx + c_2$

Where c_2 is an arbitrary constant of integration.

Hence the complete solution of (1) is given by,

$$y = uv$$

$\Rightarrow \qquad y = u \int \frac{1}{u^2} e^{-\int p\,dx}\left[\int Ru\; e^{\int p\,dx} dx + c_1\right] dx + c_2 u$

To find out the part of C.F. of the linear differential equation of II order given by

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Remember:

Sr. No.	Condition	Part of C.F.
1.	$1 + \frac{P}{a} + \frac{Q}{a^2} = 0$	e^{ax}
2.	$1 + P + Q = 0$	e^{x}
3.	$1 - P + Q = 0$	e^{-x}
4.	$m(m-1) + Pmx + Qx^2 = 0$	x^m
5.	$P + Qx = 0$	x
6.	$2 + 2Px + Qx^2 = 0$	x^2

Ques. Solve:

$$\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - (1 - \cot x)y = e^x \sin x.$$

Sol. Comparing with the standard form, we get

$$P = -\cot x, Q = -(1-\cot x), R = e^x \sin x$$

$$1 + P + Q = 1 - 1 + \cot x - \cot x = 0$$

$\therefore$ A part of $C.F. = e^x$

Let $y = ve^x$ be the complete solution of given equation, then

$$\frac{dy}{dx} = ve^x + e^x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = ve^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}$$

Substituting for $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given equation, we get

$$\frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x$$

$\Rightarrow$ $\frac{dp}{dx} + (2-\cot x)\, p = \sin x$ ...(1) where $p = \frac{dv}{dx}$.

This is a linear differential equation of I order in p and x.

$$I.F. = e^{\int (2-\cot x)dx} = \frac{e^{2x}}{\sin x}$$

Solution of (1) is, $p\frac{e^{2x}}{\sin x} = \int \sin x . \frac{e^{2x}}{\sin x} dx + c_1 = \frac{e^{2x}}{2} + c_1$

Where c_1 is an arbitrary constant of integration.

$$p = \frac{1}{2}\sin x + c_1 e^{-2x} \sin x$$

$$\frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x} \sin x$$

Integrating, we get $v = -\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(\cos x + 2\sin x) + c_2$

Hence the complete solution is given by,

$$y = v e^{x} = \left[-\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(\cos x + 2\sin x) + c_2 \right] e^{x}.$$

Reduced to Normal Form (Removal of first derivative)

Method 2. To Find the Complete Solution of $y'' + Py' + Qy = R$ when it is reduced to Normal Form (Removal of first derivative)

When the part of C.F. cannot be determined by the previous method, we reduce the given differential equation in **normal form** by eliminating the term in which there exists first derivative of the dependent variable.

$$\frac{d^2 y}{dx^2} P \frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

Let $y = uv$ be the complete solution of eqn. (1), where u and v are the function of x.

$$\therefore \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

and
$$\frac{d^2 y}{dx^2} = v\frac{d^2 u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2 v}{dx^2}$$

Substituting the value of $y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}$ in eqn. (1), we get

$$\frac{d^2 v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{du}{dx} + v\left(\frac{1}{u}\frac{d^2 u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u} \qquad \ldots(2)$$

Let us choose u such that $\frac{2}{u}\frac{du}{dx} + P = 0 \qquad \ldots(3)$

which on solving gives,

$$u = e^{-\int \frac{P}{2} dx} \qquad \ldots(4)$$

From (3), $\qquad \frac{du}{dx} = -\frac{Pu}{2}$

Differentiating, we get $\frac{d^2 u}{dx^2} = -\frac{1}{2}\left[P\left(\frac{du}{dx}\right) + \frac{dP}{dx}(u) \right]$

$$= -\frac{1}{2}\left[P\left(\frac{-Pu}{2}\right) + u\frac{dP}{dx} \right] = \frac{P^2 u}{2} - \frac{u}{2}\frac{dP}{dx}$$

Coefficient of $v = \frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q = \frac{1}{u}\left[\frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx}\right] + \frac{P}{u}\left(\frac{-Pu}{2}\right) + Q$

$$= Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = I \text{ (say)}$$

Then (2) becomes, $\frac{d^2v}{dx^2} + Iv = S$ …(5)

This is known as the normal form of equation (1).

Solving (5), we get v in terms of x. Ultimately, $y = uv$ is the complete solution.

Ques. Solve: $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x.$

Sol. Here, $P = -4x, Q = 4x^2 - 1, R = -3e^{x^2}\sin 2x$

Let $y = uv$ be the complete solution.

Now, $u = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 4x^2 - 1\frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1.$$

Also, $S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$

Hence normal form is,

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

Auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$C.F. = c_1\cos x + c_2\sin x$$

where c_1 and c_2 are arbitrary constants of integration.

$$P.I. = \frac{1}{D^2+1}(-3\sin 2x) = \frac{-3}{(-4+1)}\sin 2x = \sin 2x$$

$\therefore$ Solution is, $v = c_1 \cos x + c_2 \sin x + \sin 2x$

Hence the complete solution of given differential equation is

$$y = uv = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x) .$$

Changing the Independent Variable

Method 3. To Find the Complete Solution of $y'' + Py' + Qy = R$ by changing the Independent Variable

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

Let us relate x and z by the relation,

$$z = f(x) \qquad \ldots(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \qquad \ldots(3)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$$

$$= \frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + \frac{dz}{dx} \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + \left(\frac{dz}{dz}\right)^2 \frac{d^2y}{dx^2} \qquad \ldots(4)$$

Substituting in (1), we get

$$\frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dx^2} + P\frac{dy}{dz} \cdot \frac{dz}{dx} + Qy = R$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \ldots(5)$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}, Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}, R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Here $P_1,\ Q_1,\ R_1$ are functions of x which can be transformed into functions of z using the relation $z = f(x)$.

Choose z such that $Q_1 = \text{constant} = a^2$ (say)

$\Rightarrow \qquad \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2} a^2 \qquad \Rightarrow \qquad a\dfrac{dz}{dx} = \sqrt{Q}$

$\Rightarrow \qquad dz = \dfrac{\sqrt{Q}}{a} dx$

Integration yields, $\quad z = \int \dfrac{\sqrt{Q}}{a} dx$

If this value of z makes P_1 as constant then equation (5) can be solved.

Ques. Solve: $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol. $\quad \dfrac{d^2y}{dx^2} + \dfrac{1}{1+x}\dfrac{dy}{dx} + \dfrac{y}{(1+x)^2} = \dfrac{4}{(1+x)^2}\cos\log(1+x) \qquad \ldots(1)$

Choose z such that,

$$\left(\frac{dz}{dx}\right)^2 = \frac{1}{(1+x)^2}$$

$\Rightarrow \qquad \dfrac{dz}{dx} = \dfrac{1}{1+x} \qquad \ldots(2)$

Integration yields, $\quad z = \log(1+x) \qquad \ldots(3)$

From (2), $\quad \dfrac{d^2z}{dx^2} = -\dfrac{1}{(1+x)^2}$

$\therefore \qquad P_1 = \dfrac{-\dfrac{1}{(1+x)^2} + \dfrac{1}{1+x}\cdot\dfrac{1}{1+x}}{\dfrac{1}{(1+x)^2}} = 0$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$$

$R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2} = 4\cos\log(1+x) = 4\cos z \qquad$ (Form (3))

Reduced equation is

$$\frac{d^2y}{dz^2}+y=4\cos z$$

Auxiliary equation is $m^2+1=0$ $\Rightarrow m=\pm i$

$$C.F.=c_1\cos z+c_2\sin z$$

$$P.I.=\frac{1}{D^2+1}(4\cos z)=4.\frac{z}{2}\sin z=2z\sin z$$

Complete solution is

$$y=c_1\cos z+c_2\sin z+2z\sin z$$

$$y=c_1\cos\log(1+x)+c_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x).$$

Method of Variation of Parameters

Method4. To Find the Complete Solution of $y''+Py'+Qy=R$ by the Method of Variation of Parameters

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad \ldots(1)$$

Let the complementary function of (1) be

$$y=Au+Bv \qquad \ldots(2)$$

$\because$ u and v are part of C.F.

$$\therefore \quad \frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=0 \qquad \ldots(3)$$

and

$$\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv=0 \qquad \ldots(4)$$

Let the complete solution of (1) be

$$y=Au+Bv \qquad \ldots(5)$$

where A and B are not constants but suitable functions of x to be so chosen that (5) satisfies (1). Now,

$$y_1=Au_1+Bv_1+A_1u+B_1v$$

$$\Rightarrow \quad y_1=Au_1+Bv_1+(A_1u+B_1v) \qquad \ldots(6)$$

Let us choose A and B such that

$$A_1 u + B_1 v = 0 \quad \ldots(7)$$

Now (6) becomes, $y_1 = Au_1 + Bv_1$ …(8)

$\therefore$ $y_2 = A_1 u_1 + Au_2 + B_1 v_1 + Bv_2$ …(9)

Substituting the values of y, y_1, y_2 from (5), (8) and (9) in (1) respectively, we get

$$(A_1 u_1 + Au_2 + B_1 v_1 + Bv_2) + P(Au_1 + Bv_1) + Q(Au + Bv) = R$$

$$\Rightarrow \quad A_1 u_1 + B_1 v_1 + A(u_2 + Pu_2 + Qu) + B(v_2 + Pv_1 + Qv) = R$$

$$\Rightarrow \quad A_1 u_1 + B_1 v_1 = R \quad \ldots(10)$$ (Using (3) and (4))

Solving (7) and (10) for A_1 and B_1, we get

$$A_1 u + B_1 v = 0$$

$$A_1 u_1 + B_1 v_1 - R = 0$$

$$\Rightarrow \quad \frac{A_1}{-Rv} = \frac{B_1}{Ru} = \frac{1}{uv_1 - u_1 v}$$

$$\Rightarrow \quad A_1 = \frac{-Rv}{uv_1 - u_1 v} = \phi(x) \quad \text{(say)} \quad \ldots(11)$$

$$B_1 = \frac{Ru}{uv_1 - u_1 v} = \psi(x) \quad \text{(say)} \quad \ldots(12)$$

Integrating (11), we get $A = \int \phi(x) dx + a$ …(13)

Where a is an arbitrary constant of integration.

Integrating (12), we get $B = \int \psi(x) dx + b$ …(14)

where b is also an arbitrary constant of integration.

Putting the above values in (5), we get

$$y = \left[\int \phi(x)\,dx + a\right]u + \left[\int \psi(x)\,dx + b\right]v$$

$$\Rightarrow \quad y = u\int \phi(x)\,dx + v\int \psi(x)\,dx + au + bv$$

This gives the complete solution of (1).

Ques. Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2}+a^2y=\sec ax.$$

Sol. Here, $u=\cos ax,\ v=\sin\ ax$ are two parts of C.F.

Also, $R=\sec ax.$

Let the complete solution be

$$y=A\cos ax+B\sin ax$$

where A and B are suitable functions of x.

To determine the values of A and B, we have

$$A=\int\frac{-Rv}{uv_1-u_1v}dx+c_1$$

$$=\int\frac{-\sec ax.\sin ax}{\{\cos ax.a\cos ax-(-a\sin ax)\sin ax\}}dx+c_1$$

$$=-\int\frac{\tan ax}{a}dx+c_1$$

$$\Rightarrow \qquad A=\frac{1}{a^2}\log\cos ax+c_1$$

where c_1 is an arbitrary constant of integration.

$$B=\int\frac{Ru}{uv_1-u_1v}dx+c_2$$

$$=\int\frac{\sec ax.\cos ax}{\{\cos ax.a\cos ax-(-a\sin ax)\sin ax\}}dx+c_2$$

$$=\frac{1}{a}\int dx+c_2=\frac{x}{a}+c_2$$

where c_2 is an arbitrary constant of integration.

Hence the complete solution is given by

$$y=A\cos ax+B\sin ax$$

$$=\left(\frac{\log\cos ax}{a^2}+c_1\right)\cos ax+\left(\frac{x}{a}+c_2\right)\sin ax$$

Ques. Solve by method of variation of parameters:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{-x}\log x\,.$$

Sol. Parts of C.F. are $u=e^{-x}$, $v=xe^{-x}$ and $R=e^{-x}\log x$

Let $y=Ae^{-x}+Bxe^{-x}$ be the complete solution where A and B are some suitable functions o x. To determine A and B, we have

$$A=-\int\frac{Rv}{uv_1-u_1v}dx+c_1=-\int\frac{e^{-x}\log x.xe^{-x}}{e^{-x}(e^{-x}-xe^{-x})+xe^{-2x}}dx+c_1$$

$$=-\int x\log x\,dx+c_1=-\frac{x^2}{2}\log x+\frac{x^2}{4}+c_1$$

$$B=\int\frac{Ru}{uv_1-u_1v}dx+c_2=\int\frac{e^{-x}\log x.e^{-x}}{e^{-2x}}dx+c_2$$

$$=\int\log x\,dx+c_2=x\log x-x+c_2$$

Hence the complete solution is

$$y=Ae^{-x}+Bxe^{-x}$$

$$=\left(-\frac{x^2}{2}\log x+\frac{x^2}{4}+c_1\right)e^{-x}+(x\log x-x+c_2)xe^{-x}$$

Ques. Using variation of parameters method, solve:

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=x^3\log x.$$

Sol. Consider the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-12y=0 \text{ for finding parts of C.F.}$$

Put $x=e^z$ so that $x=\log x$ and Let $D\equiv\frac{d}{dz}$ then the given equation reduces to

$$[D(D-1)+2D-12]\,y=0$$

$\Rightarrow$ $$(D^2+D-12)\,y=0$$

Auxiliary equation is

$$m^2 + m - 12 = 0 \quad \Rightarrow \quad m = 3, -4$$

$$\therefore \qquad C.F. = c_1 e^{3z} + c_2 e^{-4z} = c_1 x^3 + c_2 x^{-4}$$

Hence, parts of C.F. are x^3 and x^{-4}

Let $y = Au + Bv$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows:

$$A = -\int \frac{Rv}{uv_1 - u_1 v}\,dx + c_1 = -\int \frac{x \log x . x^{-4}}{x^3.(-4x^{-5}) - 3x^2(x^{-4})}\,dx + c_1$$

$$= -\int \frac{x^{-3}\log x}{-7x^{-2}}\,dx + c_1 = \frac{1}{7}\int \frac{\log x}{x}\,dx + c_1 = \frac{1}{14}(\log x)^2 + c_1$$

and

$$B = \int \frac{Ru}{uv_1 - u_1 v}\,dx + c_2 = \int \frac{x \log x . x^3}{-7x^{-2}}\,dx + c_2$$

$$= -\frac{1}{7}\int x^6 \log x\,dx + c_2 = -\frac{1}{7}\left[\log x . \frac{x^7}{7} - \int \frac{1}{x}.\frac{x^7}{7}\,dx\right] + c_2$$

$$= -\frac{1}{7}\left[\frac{x^7 \log x}{7} - \frac{1}{7}\left(\frac{x^7}{7}\right)\right] + c_2 = \frac{x^7}{49}\left(\frac{1}{7} - \log x\right) + c_2$$

Hence the complete solution is given by

$$y = Ax^3 + Bx^{-4} = \left[\frac{1}{14}(\log x)^2 + c_1\right]x^3 + \left[\frac{x^7}{49}\left(\frac{1}{7} - \log x\right) + c_2\right]x^{-4}$$